

多层地基二维 Biot 固结的理论解答

The theoretical solution of two dimensional biot's consolidation of layered soils

胡亚元 王立忠 陈云敏 吴世明
(浙江大学岩土工程研究所, 杭州, 310027)

文 摘 采用积分变换、算子法研究了多层地基二维 Biot 固结问题, 获得地基表面作用荷载时任意层内任意点应力(包括孔压)、位移(包括流体流量)的一般积分形式解。按照本文的方法也能获得多层地基 Biot 固结的“Mindlin 问题”的解答。本文的结论可推广应用于边界元法求解多层二维 Biot 固结问题。

关键词 Biot 固结理论, 积分变换, 算子法, 积分解。

中图法分类号 TU 411.5

作者简介 胡亚元, 男, 1968 年生, 助理工程师, 浙江大学博士生。

Hu Yayuan Wang Lizhong Chen Yunmin Wu Shiming

(The Geotechnical Engineering Institute of Zhejiang University, Hangzhou, 310027)

Abstract Based on Biot's theory, the two dimensional consolidation problems of layered soils are studied by means of integral transform method and the arithmetic method. The general integral solutions of stress (including pore pressure) and displacement (including fluid flow) in any layer are derived with the loading acting on the surface of soils. According to the methods proposed in this paper, the consolidation solution of "Mindlin" problem in layered soils can also be achieved. The results obtained in the paper can be utilized to solve the two dimensional consolidation of layered soils by boundary method.

Key words Biot's theory of consolidation, integral transform, integral solution.

1 引 言

Biot 严格地推导了正确反映孔隙水压力消散与土骨架变形相互关系的固结理论, 一般称之为“真固结理论”^[1]。要求解 Biot 固结理论方程, 数学上十分困难^[2], R. E. Gibson, J. R. Booke 获得了有限厚地基表面沉降的复变函数固结解^[3, 4], 黄传志等在假定下卧层是刚性的硬卧层的条件下, 求得有限厚地基固结的全部解答^[5], 从文献[5]给出的固结解答的形式来看, 层内应力(包括孔压, 下同)、位移被表示成各个双重广义积分函数, 实际上是实数域内积分形式的解答。

在实际的地基中, 土体是以层状沉积的形式存在, 饱和土体通常也并不是单独一层, 因此研究多层地基的 Biot 固结的理论解答有较大的工程意义。本文采用积分变换法求解二维问题的 Biot 方程, 运用初参数、算子法解析推导各层位移(包括流体流量, 下同)、应力的传递矩阵, 再根据边界条件求得 Biot 固结问题的全部解。在推导过程中, 未引入位移函数, 简化了数学运算; 传递矩阵中包括所有的物理量, 适用于任意层任意边界条件; 采用初始函数回代方法, 对任意二维 Biot 问题, 只需求解三元一次方程即可获得地基表面的位移、应力积分表达式, 计算固结度简便; 获得的解答是一个双重广义

积分函数, 可二次采用快速傅立叶变换计算数值结果^[6, 7], 数值计算较为方便。同时, 本文的解答还为边界元法提供了多层地基 Biot 固结基本解的解析表达式。

2 推导各层地基二维 Biot 固结问题的传递矩阵

二维 Biot 固结理论可表达为:

$$\nabla^2 u_x + \frac{1}{1-2\nu} \frac{\partial e}{\partial x} - \frac{1}{G} \frac{\partial P_f}{\partial x} = 0 \quad (1a)$$

$$\nabla^2 u_z + \frac{1}{1-2\nu} \frac{\partial e}{\partial z} - \frac{1}{G} \frac{\partial P_f}{\partial z} = 0 \quad (1b)$$

$$k \nabla^2 P_f = \frac{\partial e}{\partial t} \quad (1c)$$

$$e = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \quad (1d)$$

式中 ν, G 分别为骨架的泊松比及剪切模量, u_x, u_z 为骨架水平和垂直位移, P_f 为孔隙水压力, k 为渗透系数。初始条件为

$$e|_{t=0} = 0 \quad (1e)$$

由 $\frac{\partial}{\partial x}(1a) + \frac{\partial}{\partial z}(1b)$ 得

到稿日期: 1997-05-26.

$$\alpha \nabla^2 e = \nabla P_f \quad (2)$$

式中 $\alpha = \frac{2G(1-\nu)}{1-2\nu}$ 。把式(2)代入到式(1c)得

$$C_v \nabla^2 e = \frac{\partial e}{\partial t} \quad (3)$$

上式中 $C_v = \frac{2Gk(1-\nu)}{1-2\nu}$ ，并定义算符 $\bar{\theta} = \mathbf{R}\theta = \int_{-\infty}^{\infty} \int_0^{\infty} \theta e^{-st} e^{i\xi x} dt dx$ ，把 \mathbf{R} 作用于式(1a)，(1b)，(1d)，(2)，(3)并应用(1e)得

$$\frac{i\xi}{1-2\nu} \bar{e} + \frac{d^2 \bar{u}_x}{dz^2} - \xi^2 \bar{u}_x - \frac{i\xi \bar{P}_f}{G} = 0 \quad (4a)$$

$$\frac{1}{1-2\nu} \frac{d\bar{e}}{dz} + \frac{d^2 \bar{u}_z}{dz^2} - \xi^2 \bar{u}_z - \frac{1}{G} \frac{d\bar{P}_f}{dz} = 0 \quad (4b)$$

$$\bar{e} = i\xi \bar{u}_x + \frac{d\bar{u}_z}{dz} \quad (4c)$$

$$\frac{d^2 \bar{P}_f}{dz^2} - \xi^2 \bar{P}_f = \frac{\Sigma \bar{e}}{C_v} \quad (4d)$$

$$\frac{d^2 \bar{e}}{dz^2} - q^2 \bar{e} = 0 \quad (4e)$$

其中 $a^2 = \xi^2 + \frac{s}{C_v}$ 。解上式微分方程组得到

$$\begin{aligned} i \bar{u}_x(\xi, z, s) = & -A_1 \frac{\xi C_v}{s} \text{ch} qz - A_2 \left(\frac{1}{\xi} \text{ch} \xi z + z \text{sh} \xi z \right) \\ & - A_3 \frac{1}{\xi} \text{ch} \xi z - B_1 \frac{\xi C_v}{s} \text{sh} qz \\ & - B_2 \left(\frac{1}{\xi} \text{sh} \xi z + z \text{ch} \xi z \right) - B_3 \frac{1}{\xi} \text{sh} \xi z \end{aligned} \quad (5a)$$

$$\begin{aligned} \bar{u}_z(\xi, z, s) = & A_1 \frac{q C_v}{s} \text{sh} qz + A_2 z \text{ch} \xi z + A_3 \frac{1}{\xi} \text{sh} \xi z \\ & + B_1 \frac{q C_v}{s} \text{ch} qz + B_2 z \text{sh} \xi z + B_3 \frac{1}{\xi} \text{ch} \xi z \end{aligned} \quad (5b)$$

$$\begin{aligned} \bar{P}_f(\xi, z, s) = & A_1 \alpha \text{ch} qz + 2A_2 G \text{ch} \xi z \\ & + B_1 \alpha \text{sh} qz + 2B_2 G \text{sh} \xi z \end{aligned} \quad (5c)$$

$$\bar{e}(\xi, z, s) = A_1 \text{ch} qz + B_1 \text{sh} qz \quad (5d)$$

由土骨架的本构关系及有效应力原理得

$$\sigma_{xz} = G \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \quad (6a)$$

$$\sigma'_{zz} = 2G \frac{\partial u_z}{\partial z} + 2G \frac{\nu}{1-2\nu} e \quad (6b)$$

$$\alpha_{zz} = \sigma'_{zz} - P_f \quad (6c)$$

$$Q = k \frac{\partial P_f}{\partial z} \quad (6d)$$

式(6)中 σ'_{zz} ， α_{zz} 分别是有效应力和总应力， Q 为流体单位面积的流量，算子 \mathbf{R} 作用于式(6)，再把式(5)代入得

$$i \bar{\alpha}_{xz}(\xi, Z, S) = -2G \left[A_1 \frac{\xi q C_v}{s} \text{sh} qz + A_2 (\text{sh} \xi z + \xi \text{ch} \xi z) \right.$$

$$\begin{aligned} & + A_3 \text{sh} \xi z + B_1 \frac{\xi q C_v}{s} \text{ch} qz \\ & + B_2 (\text{ch} \xi z + \xi \text{sh} \xi z) + B_3 \text{ch} \xi z \end{aligned} \quad (7a)$$

$$\begin{aligned} \bar{\alpha}_{zz}(\xi, z, s) = & 2G \left[A_1 \frac{\xi^2 C_v}{s} \text{ch} qz + A_2 \xi \text{sh} \xi z + \right. \\ & A_3 \text{ch} \xi z \\ & + B_1 \frac{\xi^2 C_v}{s} \text{sh} qz + B_2 \xi \text{sh} \xi z + B_3 \text{sh} \xi z \end{aligned} \quad (7b)$$

$$\begin{aligned} \bar{i} \bar{Q}(\xi, Z, S) = & A_1 \xi q C_v \text{sh} qz + \frac{2G \xi^2 C_v}{\alpha} A_2 \text{sh} \xi z \\ & + B_1 \xi q C_v \text{ch} qz + B_2 \frac{2G \xi^2 C_v}{\alpha} \text{ch} \xi z \end{aligned} \quad (7c)$$

为便于应用层间连续条件，把常数 $A_1, A_2, A_3, B_1, B_2, B_3$ 用边界应力、位移表示。取局部坐标 $x-z$ (全局坐标为 $X-Z$)，使该层上边界位于局部坐标 x 轴，由式(5)和式(7)取 $z=0$ 得

$$A_1 = \frac{2G\xi}{\alpha} i \bar{u}_x(\xi, 0, s) + \frac{1}{\alpha} \bar{\alpha}_{zz}(\xi, 0, s) + \frac{1}{\alpha} \bar{P}_f(\xi, 0, s) \quad (8a)$$

$$A_2 = -\xi i \bar{u}_x(\xi, 0, s) - \frac{1}{2G} \bar{P}_f(\xi, 0, s) \quad (8b)$$

$$\begin{aligned} A_3 = & \left(\frac{1}{2G} - \frac{\xi^2 C_v}{\alpha} \bar{\alpha}_{zz}(\xi, 0, s) - \frac{2G\xi^3 C_v}{\alpha} i \bar{u}_x(\xi, 0, s) \right. \\ & \left. - \frac{\xi^2 C_v}{\alpha} \bar{P}_f(\xi, 0, s) \right) \end{aligned} \quad (8c)$$

$$B_1 = \frac{\xi}{\alpha q} i \bar{\alpha}_{xz}(\xi, 0, s) + \frac{2G\xi^2}{\alpha q} \bar{u}_z(\xi, 0, s) + \frac{1}{C_v q} \bar{Q}(\xi, 0, s) \quad (8d)$$

$$B_2 = -\frac{1}{2G} i \bar{\alpha}_{xz}(\xi, 0, s) - \xi \bar{u}_z(\xi, 0, s) \quad (8e)$$

$$\begin{aligned} B_3 = & -\left(\xi - \frac{2G\xi^3 C_v}{\alpha} \right) \bar{u}_z(\xi, 0, s) - \frac{\xi^2 C_v}{\alpha} \bar{\alpha}_{xz}(\xi, 0, s) \\ & - \frac{\xi}{s} \bar{Q}(\xi, 0, s) \end{aligned} \quad (8f)$$

把式(8)回代到式(5)和式(7)并写成矩阵形式得

$$\begin{bmatrix} i \bar{u}_x(\xi, z, s) \\ \bar{u}_z(\xi, z, s) \\ \bar{\alpha}_{zz}(\xi, z, s) \\ i \bar{\alpha}_{xz}(\xi, z, s) \\ \bar{P}_f(\xi, z, s) \\ \bar{Q}(\xi, z, s) \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} & \Gamma_{26} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} & \Gamma_{35} & \Gamma_{36} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} & \Gamma_{45} & \Gamma_{46} \\ \Gamma_{51} & \Gamma_{52} & \Gamma_{53} & \Gamma_{54} & \Gamma_{55} & \Gamma_{56} \\ \Gamma_{61} & \Gamma_{62} & \Gamma_{63} & \Gamma_{64} & \Gamma_{65} & \Gamma_{66} \end{bmatrix} \begin{bmatrix} i \bar{u}_x(\xi, 0, s) \\ \bar{u}_z(\xi, 0, s) \\ \bar{\alpha}_{zz}(\xi, 0, s) \\ i \bar{\alpha}_{xz}(\xi, 0, s) \\ \bar{P}_f(\xi, 0, s) \\ \bar{Q}(\xi, 0, s) \end{bmatrix} \quad (9)$$

式中

$$\Gamma_{11} = \frac{2G\xi^2 C_v}{\alpha} (\text{ch} \xi z - \text{ch} qz) + (\text{ch} \xi z + \xi \text{sh} \xi z)$$

$$\begin{aligned} \Gamma_{12} &= \frac{2G\xi^2 C_V}{\alpha} (\text{sh } \xi z - \frac{\xi}{q} \text{sh } qz) + \xi z \text{ch } \xi z \\ \Gamma_{13} &= \frac{\xi C_V}{\alpha} (\text{ch } \xi z - \text{ch } qz) + \frac{z}{2G} \text{sh } \xi z \\ \Gamma_{14} &= \frac{\xi C_V}{\alpha} (\text{sh } \xi z - \frac{\xi}{q} \text{sh } qz) + \frac{1}{2G\xi} (\text{sh } \xi z + \xi z \text{ch } \xi z) \\ \Gamma_{15} &= \frac{\xi C_V}{\alpha} (\text{ch } \xi z - \text{ch } qz) \\ \Gamma_{16} &= \frac{1}{s} (\text{ch } \xi z - \frac{\xi}{q} \text{sh } qz) \\ \Gamma_{21} &= \frac{2G\xi C_V}{\alpha} (q \text{sh } qz - \xi \text{sh } \xi z) - \xi z \text{ch } \xi z \\ \Gamma_{22} &= \frac{2G\xi^2 C_V}{\alpha} (\text{ch } qz - \text{ch } \xi z) + (\text{ch } \xi z - \xi z \text{sh } \xi z) \\ \Gamma_{23} &= \frac{C_V}{\alpha} (a \text{sh } az - \xi \text{sh } \xi z) + \frac{1}{2G\xi} (\text{sh } \xi z - \xi z \text{ch } \xi z) \\ \Gamma_{24} &= \frac{\xi C_V}{\alpha} (\text{ch } qz - \text{ch } \xi z) - \frac{1}{2G} z \text{sh } \xi z \\ \Gamma_{25} &= \frac{C_V}{\alpha} (q \text{sh } qz - \xi \text{sh } \xi z) \\ \Gamma_{26} &= \frac{1}{s} (\text{ch } qz - \text{ch } \xi z) \\ \Gamma_{31} &= 2G\xi^2 [\frac{2G\xi C_V}{\alpha} (\text{ch } qz - \text{ch } \xi z) - z \text{sh } \xi z] \\ \Gamma_{32} &= 2G\xi [\frac{2G\xi^2 C_V}{\alpha} (\frac{\xi}{q} \text{sh } qz - \text{sh } \xi z) + \xi (\text{sh } \xi z - \xi z \text{ch } \xi z)] \\ \Gamma_{33} &= 2G [\frac{\xi^2 C_V}{\alpha} (\text{ch } qz - \text{ch } \xi z) + \frac{1}{2G} (\text{ch } \xi z - \xi z \text{sh } \xi z)] \\ \Gamma_{34} &= 2G [\frac{\xi^2 C_V}{\alpha} (\frac{\xi}{q} \text{sh } qz - \text{sh } \xi z) - \frac{1}{2G} \xi z \text{ch } \xi z] \\ \Gamma_{35} &= 2G [\frac{\xi^2 C_V}{\alpha} (\text{ch } qz - \text{ch } \xi z)] \\ \Gamma_{36} &= 2G [\frac{\xi}{s} (\frac{\xi}{q} \text{sh } az - \text{sh } \xi z)] \\ \Gamma_{41} &= 2G [\frac{2G\xi^2 C_V}{\alpha} (\xi \text{sh } \xi z - q \text{sh } qz) + \xi (\text{sh } \xi z + \xi z \text{ch } \xi z)] \\ \Gamma_{42} &= 2G\xi^2 [\frac{2G\xi C_V}{\alpha} (\text{ch } \xi z - \text{ch } qz) + z \text{sh } \xi z] \\ \Gamma_{43} &= 2G [\frac{\xi C_V}{\alpha} (\xi \text{sh } \xi z - q \text{sh } qz) + \frac{1}{2G} \xi z \text{ch } \xi z] \\ \Gamma_{44} &= 2G [\frac{\xi^2 C_V}{\alpha} (\text{ch } \xi z - \text{ch } qz) + \frac{1}{2G} (\text{ch } \xi z + \xi z \text{sh } \xi z)] \\ \Gamma_{45} &= 2G \frac{\xi C_V}{\alpha} (\xi \text{sh } \xi z - q \text{sh } qz) \\ \Gamma_{46} &= 2G \frac{\xi}{s} (\text{ch } \xi z - \text{ch } qz) \\ \Gamma_{51} &= 2G\xi (\text{ch } qz - \text{ch } \xi z) \\ \Gamma_{52} &= 2G\xi (\frac{\xi}{q} \text{sh } qz - \text{sh } \xi z) \\ \Gamma_{53} &= \text{ch } qz - \text{ch } \xi z \end{aligned}$$

$$\begin{aligned} \Gamma_{54} &= \frac{\xi}{q} \text{sh } qz - \text{sh } \xi z \\ \Gamma_{55} &= \text{ch } qz \\ \Gamma_{56} &= \frac{\alpha}{qG_V} \text{sh } qz \\ \Gamma_{61} &= \frac{2G\xi C_V}{\alpha} (q \text{sh } qz - \xi \text{sh } \xi z) \\ \Gamma_{62} &= \frac{2G\xi^2 C_V}{\alpha} (\text{ch } qz - \text{ch } \xi z) \\ \Gamma_{63} &= \frac{C_V}{\alpha} (q \text{sh } qz - \xi \text{sh } \xi z) \\ \Gamma_{64} &= \frac{\xi C_V}{\alpha} (\text{ch } qz - \text{ch } \xi z) \\ \Gamma_{65} &= \frac{qC_V}{\alpha} \text{sh } qz \\ \Gamma_{66} &= \text{ch } qz \end{aligned}$$

记

$$\begin{aligned} \bar{Y}(\xi, z, s) &= [i \bar{u}_x(\xi, z, s), \bar{u}_z(\xi, z, s), \bar{\sigma}_{zz}(\xi, z, s), \\ &\quad i \bar{\sigma}_{xz}(\xi, z, s), \bar{P}_f(\xi, z, s), \bar{Q}(\xi, z, s)]^T \\ \bar{Y}(\xi, 0, s) &= [i \bar{u}_x(\xi, 0, s), \bar{u}_z(\xi, 0, s), \bar{\sigma}_{zz}(\xi, 0, s), \\ &\quad i \bar{\sigma}_{xz}(\xi, 0, s), \bar{P}_f(\xi, 0, s), \bar{Q}(\xi, 0, s)]^T \\ [\Phi(\xi, z, s)] &= [\Gamma_{ij}(\xi, z, s)]_{6 \times 6}, \text{ 则式(9)可写为:} \\ \bar{Y}(\xi, z, s) &= [\Phi(\xi, z, s)] \bar{Y}(\xi, 0, s) \quad (10) \end{aligned}$$

3 多层地基 Biot 固结的传递矩阵解法

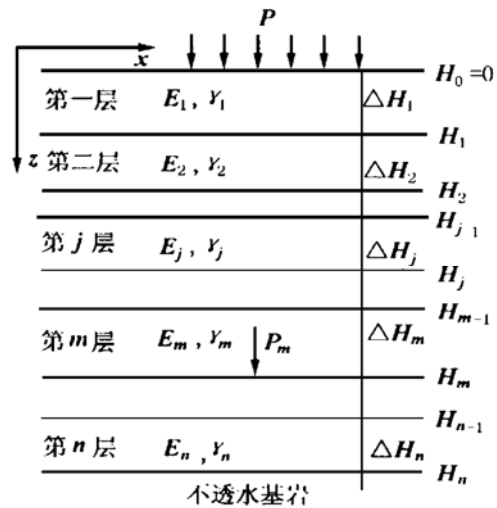


图 1 计算简图

Fig. 1 Simple model for calculation

图 1 为一 n 层地基, 在地基表面有荷载 $P(x, t)$, 在第 m 层也有荷载 $P_m(x, t)$ ($m < n$), (把力作用点所在平面也作为一分界层, 因为应力产生突变), 对于任意的第 j 层 ($j \leq n$), 局部坐标和全局坐标的关系为 $z = Z - H_{j-1}$, 层厚 $\Delta H_j = H_j - H_{j-1}$, 则由式(10)得

$$\bar{Y}(\xi, H_j^-, s) = [\Phi(\xi, \Delta H_j, s)] \bar{Y}(\xi, H_{j-1}^+, s) \quad (11)$$

上式的 $\bar{Y}(\xi, H_j^-, s)$, $\bar{Y}(\xi, H_{j-1}^+, s)$ 分别是第 j 层底

面和顶面的物理量组成的一维矩阵。令 j 从 1 取到 n , 由式(11)得

$$\overline{Y}(\zeta, H_1^-, s) = [\Phi_1(\zeta, \Delta H_1, s)] \overline{Y}(\zeta, 0, s) \quad (12a)$$

$$\overline{Y}(\zeta, H_2^-, s) = [\Phi_2(\zeta, \Delta H_2, s)] \overline{Y}(\zeta, H_1^+, s) \quad (12b)$$

$$\overline{Y}(\zeta, H_j^-, s) = [\Phi_j(\zeta, \Delta H_j, s)] \overline{Y}(\zeta, H_{j-1}^+, s) \quad (12c)$$

$$\overline{Y}(\zeta, H_n^-, s) = [\Phi_n(\zeta, \Delta H_n, s)] \overline{Y}(\zeta, H_{n-1}^+, s) \quad (12d)$$

根据界面处的应力和位移连续可知

$$\begin{pmatrix} i u_x(X, H_j^+, t) \\ u_z(X, H_j^+, t) \\ \sigma_{zz}(X, H_j^+, t) \\ i \sigma_{xz}(X, H_j^+, t) \\ P_f(X, H_j^+, t) \\ Q(X, H_j^+, t) \end{pmatrix} = \begin{pmatrix} i u_x(X, H_j^-, t) \\ u_z(X, H_j^-, t) \\ \sigma_{zz}(X, H_j^-, t) \\ i \sigma_{xz}(X, H_j^-, t) \\ P_f(X, H_j^-, t) \\ Q(X, H_j^-, t) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \times P_m(x, t) \quad (13)$$

对式(13)作用算子 \mathbf{R} 得

$$\begin{pmatrix} i \overline{u}_x(X, H_j^+, s) \\ \overline{u}_z(X, H_j^+, s) \\ \overline{\sigma}_{zz}(X, H_j^+, s) \\ i \overline{\sigma}_{xz}(X, H_j^+, s) \\ \overline{P}_f(X, H_j^+, s) \\ \overline{Q}(X, H_j^+, s) \end{pmatrix} = \begin{pmatrix} i \overline{u}_x(X, H_j^-, s) \\ \overline{u}_z(X, H_j^-, s) \\ \overline{\sigma}_{zz}(X, H_j^-, s) \\ i \overline{\sigma}_{xz}(X, H_j^-, s) \\ \overline{P}_f(X, H_j^-, s) \\ \overline{Q}(X, H_j^-, s) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \overline{P}_m(\zeta, s) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (14)$$

应用式(12) 式(14)从底面依次向上进行递推得

$$\overline{Y}(\zeta, H_n^-, s) = [\Lambda] \overline{Y}(\zeta, 0, s) - [E] [0, 0, \overline{P}_m(\zeta, s), 0, 0, 0]^T \quad (15)$$

式中 $[\Lambda] = [\Phi_n][\Phi_{n-1}] \dots [\Phi_2][\Phi_1]$, $[E] = [\Phi_n][\Phi_{n-1}] \dots [\Phi_{m+1}]$ 。式(15)共有十二个物理量(表面六个, 最底层六个), 六个方程, 因此, 根据边界条件补充另六个方程可唯一地求解式(15)。同时, 式(15)中包含地基内部作用荷载的情况, 因此能计算多层地基二维 Biot 固结的“Mindlin”问题。不失一般性, 假设边界条件如图 1 所示, 饱和土层内无荷载, 则用算子 \mathbf{R} 作用于顶面边界条件得

$$\begin{pmatrix} \overline{\sigma}_{zz}(\zeta, 0, s) = -\overline{P}(\zeta, s) \\ \overline{P}_f(\zeta, 0, s) = 0 \\ \overline{\sigma}_{xz}(\zeta, 0, s) = 0 \end{pmatrix} \quad (16a)$$

用算子 \mathbf{R} 作用于底面边界条件得

$$\begin{pmatrix} \overline{u}_x(\zeta, H_n^-, s) = 0 \\ \overline{u}_z(\zeta, H_n^-, s) = 0 \\ \overline{Q}(\zeta, H_n^-, s) = 0 \end{pmatrix} \quad (16b)$$

把以上条件代入到式(15), 由矩阵分块求解得表面的另三个物理量为

$$\{\overline{C}(\zeta, 0, s)\} = [T] \{\overline{D}(\zeta, 0, s)\} \quad (17)$$

其中

$$\{\overline{C}(\zeta, 0, s)\} = [i \overline{u}_x(\zeta, 0, s), \overline{u}_z(\zeta, 0, s), \overline{Q}(\zeta, 0, s)]^T$$

$$\{\overline{D}(\zeta, 0, s)\} = [-\overline{P}(\zeta, s), 0, 0]^T$$

$$[T] = -[\Lambda_I]^{-1}[\Lambda_{II}]$$

$$[\Lambda_I] = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{16} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{26} \\ \Lambda_{61} & \Lambda_{62} & \Lambda_{66} \end{pmatrix}$$

$$[\Lambda_{II}] = \begin{pmatrix} \Lambda_{13} & \Lambda_{14} & \Lambda_{15} \\ \Lambda_{23} & \Lambda_{24} & \Lambda_{25} \\ \Lambda_{64} & \Lambda_{64} & \Lambda_{65} \end{pmatrix}$$

同理可计算出最底层的另三个未知的物理量。

由边界条件及式(17), 地基表面所有的应力、位移都已知, 多层饱和土体的任一点可由地基表面位移、应力递推得出。对于多层饱和土体中任意第 j 层内任一点, 其应力和位移的解答由式(10)及局部坐标和全局坐标的关系得

$$\overline{Y}(\zeta, Z, s) = [\Phi(\zeta, Z - H_{j-1}, s)] \overline{Y}(\zeta, H_{j-1}^+, s) \quad (18)$$

由层间递推关系式(12) 式(14)向上递推得

$$\begin{aligned} \overline{Y}(\zeta, Z, s) &= [\Phi(\zeta, Z - H_{j-1}, s)] [\Phi_{-1}(\zeta, \Delta H_{j-1}, s)] \\ &\quad \dots [\Phi_2(\zeta, \Delta H_2, s)] [\Phi_1(\zeta, \Delta H_1, s)] \overline{Y}(\zeta, 0, s) \end{aligned} \quad (19)$$

把式(17)代入到式(19), 并相应地对算子 \mathbf{R} 进行逆变换得

$$\begin{pmatrix} i u_x(X, Z, t) \\ u_z(X, Z, t) \\ \sigma_{zz}(X, Z, t) \\ i \sigma_{xz}(X, Z, t) \\ P_f(X, Z, t) \\ Q(X, Z, t) \end{pmatrix} = \frac{1}{4\pi^2 i} \int_{a-i\infty}^{a+i\infty} \int_{-\infty}^{\infty} [\Phi_{Z=Z-h_{j-1}}] [\Phi_{-2}] \dots$$

$$[\Phi_2][\Phi_1] \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ T_{31} & T_{32} & T_{33} \end{pmatrix} - \begin{pmatrix} \overline{P}(\zeta, s) \\ 0 \\ 0 \end{pmatrix} e^{i\zeta x} e^{st} d\zeta ds \quad (20)$$

4 解答的数值计算方法

式(20)是二维多层 Biot 固结问题的双重广义积分形式解,一般必须采用数值反变换求积分解答^[6,7];本文采用双重快速傅立叶变换计算数值结果, Biot 理论固结度定义为

$$U = (u_z - u_z|_{t=0}) / (u_z|_{t=\infty} - u_z|_{t=0}) \quad (21)$$

图2曲线1所示的是地表作用宽度为 $B = 0.5H_2$ 的均布荷载时固结度随时间的变化图,其变化规律与文献[5]相同,曲线2为双层地层时固结度随时间的变化曲线。其中 $T_v = C_{v1}t/H_2^2$, C_{v1} 为上层的 C_v 。

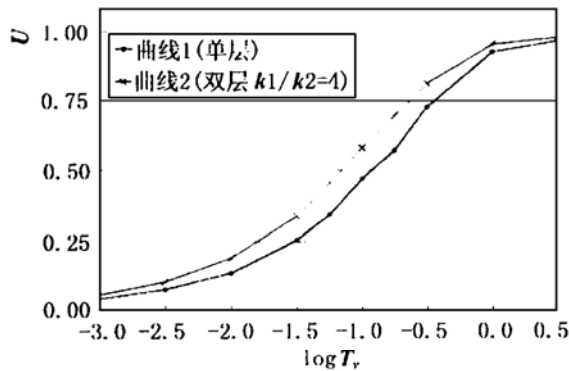


图2 固结度随时间变化图

Fig. 2 Degree of consolidation varying with time

5 结 语

本文采用积分变换,利用算子回代法及传递矩阵技术研究了任意多层饱和土体二维 Biot 固结的积分形式解。本文针对 Biot 固结方程,首先建立了任意层内边界应力,位移的传递矩阵,基于此,利用矩阵递推

法及顶底两层的边界条件获得了任意多层任意点二维 Biot 固结的应力、位移一般解析表达式。在推导过程中,不但考虑了地基表面的垂直作用力情形,而且提供了考虑地基内部垂直作用力时的解题方法,因此也可以求解 Biot 固结的“Mindlin”问题。由于 Biot 固结的固结度定义为地基表面沉降的函数,由本文的解答可直接计算地面沉降,而无须求繁杂的联立方程,计算多层地基的固结度非常方便;文末的算例分析了在地表面作用均布荷载时单双层地基固结度随时间的变化规律。众所周知,构造边界元必须运用基本解的解析表达式,本文的方法可进一步推广到用边界元求解多层饱和地基二维 Biot 固结问题。

参 考 文 献

- 1 Biot M A . General theory of three dimensional consolidation. *J Appl Phys* , 1941, **12**: 155~ 164.
- 2 黄文熙主编. 土的工程性质. 北京: 水利电力出版社. 1983. 168~ 179.
- 3 Gibson R E , Schiffman R L and Pu S L . Plane strain and axially symmetric consolidation of a claylayer on a smooth impervious base. *J Mech Appl Math*, 1970, **23**: 550.
- 4 Booker J R . The consolidation of a finite layer subject to surface loading. *Int J Solid Structure*, 1974, **10**: 1053~ 1065.
- 5 黄传志, 肖原. 二维固结问题的解析解. *岩土工程学报*, 1996, **18**(3): 47~ 54.
- 6 Durbin F. Numerical inversion of Laplace transforms: an efficient improvement to Dubner and Abacté's method. *Comput J*, 1974, **17**: 371~ 376.
- 7 Sneddon L N. *Fourier transform*. New York: McGraw-Hill, 1951.